

## Line junction in a quantum Hall system with two filling fractions

Diptiman Sen and Amit Agarwal

*Center for High Energy Physics, Indian Institute of Science, Bangalore 560 012, India*

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We present a microscopic model for a line junction formed by counter or copropagating single mode quantum Hall edges corresponding to different filling factors. The ends of the line junction can be described by two possible current splitting matrices which are dictated by the conditions of both lack of dissipation and the existence of a linear relation between the bosonic fields. Tunneling between the two edges of the line junction then leads to a microscopic understanding of a phenomenological description of line junctions introduced some time ago. The effect of density-density interactions between the two edges is considered, and renormalization-group ideas are used to study how the tunneling parameter changes with the length scale. This leads to a power-law variation of the conductance of the line junction with the temperature. Depending on the strength of the interactions the line junction can exhibit two quite different behaviors. Our results can be tested in bent quantum Hall systems fabricated recently.

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### I. INTRODUCTION

The recent fabrication of quantum Hall (QH) systems which have a sharp bend of  $90^\circ$  provides a new arena for testing theories of quantum Hall edge states.<sup>1,2</sup> By applying an appropriately tilted magnetic field, one can create a situation in which the two faces of the bent system are both in QH states but they have different filling fractions  $\nu_1$  and  $\nu_2$ ; this is because the filling fractions are governed by the components of the magnetic field perpendicular to the faces. If the magnetic field is sufficiently tilted, the two perpendicular components can even have opposite signs. Depending on whether  $\nu_1$  and  $\nu_2$  have the same sign or opposite signs, the edge states on the two sides of the line which separates the two QH states (called a line junction) propagate in opposite directions or in the same direction; these are called counter or copropagating edge states, respectively. Note that in a QH system, currents flow only along the edges because all states in the bulk are localized.

In a fractional QH system in which the two sides have the same filling fraction, the properties of a line junction (LJ) have been studied extensively;<sup>3-11</sup> they are known to provide a realization of a one-dimensional system of spinless interacting electrons with a tunable Luttinger parameter.<sup>12-14</sup> A LJ in such a system can be formed by creating a narrow barrier which divides a QH liquid such that there are chiral edge states flowing on the two sides of the barrier.<sup>15-19</sup> In general the edges interact with each other through a short-range (screened Coulomb) repulsion. A LJ is therefore similar to a nonchiral quantum wire; however, the physical separation between the two edges of the effective nonchiral wire can be controlled by a gate voltage which allows for a greater degree of control over the strength of the interaction between the edges.

It is known that a LJ can be disordered, so that the tunneling amplitude across the LJ can be taken to be a random variable. The disorder can drive a localization-delocalization transition;<sup>5</sup> the scaling dimension of the tunneling operator and therefore the occurrence of the transition generally depends on the strength of the density-density interaction.

Novel metallic and insulating states have been observed for a LJ in a bent QH system in which the filling fraction is the same on the two sides.<sup>2</sup> The results of Ref. 5 have been used to understand these states. It would clearly be interesting to extend this analysis to the case in which the two sides of the LJ have different filling fractions for which experimental results are expected to be available in the near future. (A system in which one side is an electron-gas reservoir, so that  $\nu_1=1$ , and the other side is a fractional QH state with  $\nu_2 < 1$  has been studied in Ref. 20. A similar system has been studied in the context of QH edge reconstruction<sup>21</sup>).

In this paper, we develop a microscopic model for a LJ between two QH states with different filling fractions; our model will combine ideas from several earlier papers. For reasons discussed below, we will work in the regime where the thermal decoherence length  $L_T$  is much smaller than both the length  $L$  of the LJ and the scattering mean free path  $L_m$ . We consider simple quantum Hall states on the two sides of the LJ so that the each edge consists of only one chiral mode; this will happen if the filling fractions on the two sides  $\nu_1, \nu_2$  are given by the inverses of odd integers such as  $1, 3, 5, \dots$ . In Sec. II, we discuss the idea of a current splitting matrix  $S$  for a system with two incoming and two outgoing edges. On general grounds, such a matrix is described by a single parameter  $t$  called the scattering coefficient; this parameter was phenomenologically introduced in Refs. 22 and 23. The main aim of our work will be to provide a microscopic model for the origin of the parameter  $t$ , and then to understand how  $t$  varies with the length scale or the temperature. The microscopic model will be developed in two stages. First, in Sec. III, we introduce a current splitting matrix  $S$  at each end of the LJ (described by the points  $x=0$  and  $x=L$ ). We show that the requirement that the current splitting matrix should not lead to any dissipation at the ends of the LJ yields only two possibilities for the matrix  $S$ ; the forms of  $S$  depend on the values of  $\nu_1, \nu_2$ . It turns out, interestingly, that exactly the same two possibilities for  $S$  arise if we demand that the bosonic fields describing the chiral edge modes should be related to each other in a linear way. Next, in Sec. IV, we introduce the possibility of tunneling from a point on one edge of the LJ to the corresponding point on the other edge;

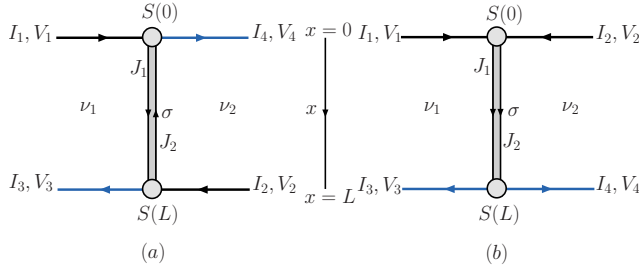


FIG. 1. (Color online) Schematic of a line junction with (a) counter-propagating and (b) copropagating modes.

this is described by a tunneling conductance per unit length  $\sigma$  which can depend on the location of the tunneling point  $x$ . (We assume that the electrons equilibrate after tunneling; this requires a mechanism for power dissipation in the LJ). For  $L_T \ll L_m$ , a kinetic equation description<sup>24</sup> of tunneling leads to a combined current splitting matrix  $S_{LJ}$  for the LJ as a whole which depends on  $\sigma$ ,  $L$  and the  $S$  matrices at the two ends of the LJ. We then turn to the temperature dependence of  $S_{LJ}$  in Sec. V. We consider density-density interactions between the two edges of the LJ and allow for tunneling with a random strength between the edges. We then use renormalization-group (RG) ideas to study how  $\sigma$  varies with the temperature.<sup>5</sup> In a certain regime ( $L_T \ll L_m \ll L$ ), the variation turns out to be given by a power law, where the power depends on the strength of the interaction between the two edges. Finally, the conductance of the LJ can be related to the matrix  $S_{LJ}$ . This combination of ideas thus gives a complete microscopic understanding of the conductance of the LJ, including its dependence on the temperature and length. In Sec. VI, we discuss how our results can be experimentally tested in QH systems with two different filling fractions. We summarize our results and discuss possible extensions of our work in Sec. VII.

## II. CURRENT SPLITTING MATRIX

The main aim of our work will be to develop a model for the current splitting matrix for a system with a line junction. To see what this matrix means, consider the systems shown in Fig. 1. In both the systems, the currents (voltages) in the two incoming edges are denoted as  $I_1$  ( $V_1$ ) and  $I_2$  ( $V_2$ ), while the currents (voltages) in the two outgoing edges are denoted as  $I_3$  ( $V_3$ ) and  $I_4$  ( $V_4$ ). Here 1 and 3 denote the edges of a QH system with filling fraction  $\nu_1$ , while 2 and 4 denote the edges of a system with filling fraction  $\nu_2$ . In the linear-response regime and assuming equilibration of the electrons, the currents and voltages on a QH edge are related as

$$I_i = \frac{e^2}{h} \nu_i V_i. \quad (1)$$

We expect that the outgoing currents should be related to the incoming ones by a real matrix denoted as  $S_{LJ}$ ,

$$\begin{pmatrix} I_3 \\ I_4 \end{pmatrix} = S_{LJ} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}. \quad (2)$$

This relation must be consistent with two general conditions:

(i) current conservation, which implies that each column of  $S$  should add up to 1, and

(ii) if the incoming voltages  $V_1$  and  $V_2$  are equal to each other, the outgoing voltages  $V_3$  and  $V_4$  should be equal to the same quantity.

Combined with Eq. (1), these two conditions allow a general current splitting matrix of the form

$$S_{LJ} = \begin{pmatrix} 1 - \frac{2t\nu_2}{\nu_1 + \nu_2} & \frac{2t\nu_1}{\nu_1 + \nu_2} \\ \frac{2t\nu_2}{\nu_1 + \nu_2} & 1 - \frac{2t\nu_1}{\nu_1 + \nu_2} \end{pmatrix}, \quad (3)$$

where the real parameter  $t$  is called the scattering coefficient;<sup>22,23</sup>  $t=0$  represents minimum tunneling and  $t=1$  maximum tunneling between the two QH fluids.

Next, we consider the power dissipated by the system. This is given by the difference of the incoming and outgoing power, namely,

$$P = \frac{1}{2}(I_1 V_1 + I_2 V_2 - I_3 V_3 - I_4 V_4) = \frac{e^2}{2h}(\nu_1 V_1^2 + \nu_2 V_2^2 - \nu_1 V_3^2 - \nu_2 V_4^2) = \frac{e^2}{h} \frac{2\nu_1 \nu_2}{\nu_1 + \nu_2} t(1-t)(V_1 - V_2)^2. \quad (4)$$

The condition that  $P \geq 0$  requires that  $0 \leq t \leq 1$ ; no power is dissipated if  $t=0$  or 1. For any value of  $\nu_1, \nu_2$  and a given voltage difference  $V_1 - V_2$ , the maximum power is dissipated when  $t=1/2$ . Curiously, we note that  $\det(S_{LJ})=1-2t$ , and vanishes at  $t=1/2$ . The reason for power dissipation in our model will be discussed in Sec. IV.

Using Eq. (1), we can rewrite Eqs. (2) and (3) as

$$V_3 - V_1 = \frac{2t\nu_2}{\nu_1 + \nu_2}(V_2 - V_1),$$

$$V_4 - V_2 = \frac{2t\nu_1}{\nu_1 + \nu_2}(V_1 - V_2). \quad (5)$$

If  $\nu_1 \neq \nu_2$  and  $t$  lies in the range  $(\nu_1 + \nu_2)/(2 \max(\nu_1, \nu_2)) < t < 1$ , we see that  $V_3$  or  $V_4$  can be higher than  $\max(V_1, V_2)$  or lower than  $\min(V_1, V_2)$ . The system can therefore act as a dc step-up transformer.<sup>22,23</sup>

## III. END OF A LINE JUNCTION

In this section, we will consider a current splitting matrix for each end of the LJ. The reason the ends of the LJ (the points  $x=0$  and  $L$  in Fig. 1) have to be treated in a special way is that the edges of two different quantum Hall systems are coming together there to form the line junction. We therefore need to impose some boundary conditions at those points. Since we will use the technique of bosonization, we will impose certain linear relations between the bosonic fields as discussed below.

Each end of the LJ is a point where four edges meet, two of them incoming and two outgoing. One incoming and one outgoing edge is associated with a filling fraction  $\nu_1$  and the other incoming and outgoing edge is associated with filling

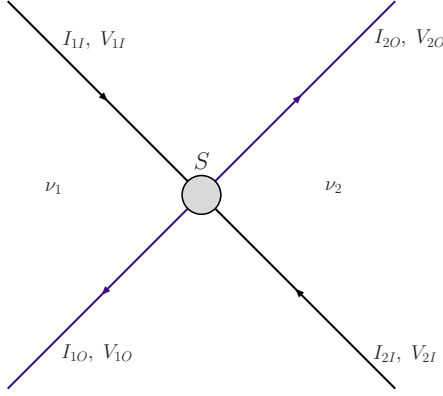


FIG. 2. (Color online) A meeting point of two incoming and two outgoing edges corresponding to two different QH filling fractions.

fraction  $\nu_2$  as shown in Fig. 2. For simple filling fractions  $\nu_i$  given by the inverse of an odd integer, each edge is associated with a single chiral boson as follows. Taking the coordinate on an edge to go from  $x=0$  to  $x=\infty$  ( $-\infty$ ) for an outgoing (incoming) edge, respectively, the Lagrangian is given by

$$\mathcal{L} = \sum_{i=1}^2 \left[ \frac{1}{4\pi\nu_i} \int_0^\infty dx \partial_x \phi_{iO} (-\partial_t - \nu_i \partial_x) \phi_{iO} + \frac{1}{4\pi\nu_i} \int_{-\infty}^0 dx \partial_x \phi_{iI} (-\partial_t - \nu_i \partial_x) \phi_{iI} \right], \quad (6)$$

where  $i$  labels the wire,  $\nu_i$  denotes the velocity, and the outgoing (incoming) fields are denoted as  $\phi_{iI}$  ( $\phi_{iO}$ ), respectively.

If  $\nu_i$  is the filling fraction associated with edge  $i$ , the quasielectron and electron annihilation operators are given by  $\psi_{i,qe} \sim \eta_i e^{i\phi_i}$  and  $\psi_{i,el} \sim \chi_i e^{i\phi_i/\nu_i}$ , respectively, where  $\eta_i$  and  $\chi_i$  are the Klein factors for quasielectrons and electrons, respectively. The density fields canonically conjugate to  $\phi_{iO/I}$  are given by  $\rho_{iO/I} = (1/2\pi) \partial_x \phi_{iO/I}$ , so that

$$[\rho_{iO/I}(x), \rho_{jO/I}(y)] = -i \delta_{ij} \frac{\nu_i}{2\pi} \partial_x \delta(x-y) \quad (7)$$

for points  $x, y$  both lying on outgoing (incoming) edges labeled  $i, j$ . On the outgoing (incoming) edge  $i$ , the outgoing (incoming) current is given by  $j_{iO/I} = -(1/2\pi) \partial_t \phi_{iO/I}$ . Hence current conservation implies that  $\sum_{i=1}^2 (\phi_{iO} - \phi_{iI})_{x=0} = 0$ . This implies that the fields must be related at  $x=0$  as

$$\begin{pmatrix} \phi_{1O} \\ \phi_{2O} \end{pmatrix} = S \begin{pmatrix} \phi_{1I} \\ \phi_{2I} \end{pmatrix}, \quad (8)$$

where the current splitting matrix  $S$  is real and each of its columns add up to 1.

Let us now decompose the fields at time  $t=0$  as

$$\phi_{iO} = \int_0^\infty \frac{dk}{k} (b_{iOk} e^{ikx} + b_{iOk}^\dagger e^{-ikx}),$$

and

$$\phi_{iI} = \int_0^\infty \frac{dk}{k} (b_{iIk} e^{ikx} + b_{iIk}^\dagger e^{-ikx}), \quad (9)$$

where the bosonic creation and annihilation operators must satisfy the commutation relations

$$[b_{iOk}, b_{jOk'}^\dagger] = [b_{iIk}, b_{jIk'}^\dagger] = \delta_{ij} \nu_i k \delta(k-k') \quad (10)$$

in order to satisfy Eq. (7). If we now demand that the commutation relation in Eq. (10) must be consistent with the relation in Eq. (8), we see that the matrix  $S$  must satisfy

$$S \begin{pmatrix} \nu_1 & 0 \\ 0 & \nu_2 \end{pmatrix} S^T = \begin{pmatrix} \nu_1 & 0 \\ 0 & \nu_2 \end{pmatrix}. \quad (11)$$

Using the condition of current conservation, namely, that the columns of  $S$  should add up to 1, we find that Eq. (11) implies that  $S$  can only take two possible values, namely,

$$S_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and

$$S_1 = \frac{1}{\nu_1 + \nu_2} \begin{pmatrix} \nu_1 - \nu_2 & 2\nu_1 \\ 2\nu_2 & \nu_2 - \nu_1 \end{pmatrix}. \quad (12)$$

Note that both these matrices satisfy  $S^2 = I$ . For the special case of  $\nu_1 = \nu_2$ , the second matrix reduces to

$$S_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The power dissipated at the point  $x=0$  is given by the difference of the incoming power  $(1/2)(I_{1I}V_{1I} + I_{2I}V_{2I})$  and the outgoing power  $(1/2)(I_{1O}V_{1O} + I_{2O}V_{2O})$ . Using Eq. (1), we find that the condition that no power is dissipated at  $x=0$  is equivalent, in terms of the bosonic fields, to the relation  $\sum_{i=1}^2 (\phi_{iI}^2/\nu_i - \phi_{iO}^2/\nu_i) = 0$ . This implies that

$$S^T \begin{pmatrix} 1/\nu_1 & 0 \\ 0 & 1/\nu_2 \end{pmatrix} S = \begin{pmatrix} 1/\nu_1 & 0 \\ 0 & 1/\nu_2 \end{pmatrix}. \quad (13)$$

This is the same condition as Eq. (11) since  $S^2 = I$ , and we therefore obtain the same solutions as in Eq. (12). We thus see that the conditions of zero power dissipation and a linear relation between the bosonic fields at the point  $x=0$  are equivalent to each other; both of them imply that the variable  $t$  appearing in the current splitting matrix [Eq. (3)] at the point  $x=0$  must be equal to 0 or 1.

The reason that the conditions of zero power dissipation and the existence of a linear relation between the bosonic fields at  $x=0$  are equivalent to each other is that both of these are consistent with a hermitian Hamiltonian and unitary evolution for the bosonic fields. In contrast to this, the discussion in Sec. IV will involve tunneling of electrons from one edge to another everywhere on the line junction away from the end points; the tunneling will be assumed to be followed by equilibration of the edge modes, and therefore some power must be dissipated. Hence we do not expect a current splitting matrix satisfying Eq. (11) to appear at any point in the line junction away from the ends.

In Sec. IV, we will consider a LJ with either counter or copropagating edges as shown in Fig. 1. We will assume that each end of the LJ (i.e., the points at  $x=0$  and  $L$ ) is associated with one of the matrices  $S_0$  or  $S_1$  given in Eq. (12); there are therefore four different possibilities for the two ends taken together. Whether one should introduce the matrix  $S_0$  or  $S_1$  at each end of the LJ depends on the physical situation at that end. If there is a large potential barrier there which widely separates the QE fluids with filling fractions  $\nu_1$  and  $\nu_2$  (and therefore minimizes the possibility of tunneling), or equivalently, if the edges meet at the end with sharp boundaries, we should choose the matrix  $S_0$ . On the other hand, if the edges meet at a point with adiabatic (smooth) boundaries (with the two quantum Hall fluids having a greater degree of contact which allows for larger tunneling), then the matrix  $S_1$  should be chosen.<sup>23</sup>

#### IV. KINETIC EQUATION APPROACH

We now study what happens inside the LJ away from the ends. We will do this using a simple kinetic equation approach.<sup>24</sup> We assume that we are in a steady state and there is local equilibrium at each point  $x$  of the LJ. In a steady state, the density  $\rho_i(x)$  is independent of time at each point  $x$ ; here  $i=1, 2$  denotes the edges on the two sides of the LJ. By the equation of continuity, the currents on the two edges  $J_1(x)$  and  $J_2(x)$  can change with  $x$  only if there is a current flow from one edge to the other. If there is a tunneling conductance per unit length given by  $\sigma(x)$ , the current flow from one edge to the other is given by  $\sigma(x)$  multiplied by the potential difference between the two edges at the point  $x$ . Assuming local equilibrium, the potential at any point of a QH edge is related to the current as  $V(x)=(h/\nu e^2)J(x)$ . We thus obtain a differential equation for the currents  $J_1(x)$  and  $J_2(x)$ . To solve this equation, it will be convenient to separately discuss the cases of LJs with counter and copropagating modes shown in Figs. 1(a) and 1(b), respectively.

Before proceeding further, we note that the tunneling between the edges lying on the two sides of the line junction leads to power dissipation. This is because the two points between which the electrons are tunneling are at different potentials. We are implicitly assuming that the electrons equilibrate after tunneling; this requires a dissipative mechanism within the line junction. Detailed discussions of equilibration in quantum Hall systems have been presented in Refs. 24 and 25.

##### A. Counter propagating modes

In the situation shown in Fig. 1(a), we find that the currents  $J_i(x)$  satisfy the equations

$$\partial_x J_1 = \partial_x J_2 = \frac{\sigma h}{e^2} \left( \frac{J_2}{\nu_2} - \frac{J_1}{\nu_1} \right). \quad (14)$$

Note that  $J_1(x) - J_2(x)$  is constant along the LJ as one expects by current conservation. If we assume that  $\sigma$  is independent of  $x$ , we can solve the above equations to obtain

TABLE I. The scattering coefficient  $t$  for the four possible choices of  $S$  matrices at the ends of the LJ, for  $\nu_1 \neq \nu_2$ .

$S(0)$	$S(L)$	$t$	$t(L/l_c \rightarrow 0)$	$t(L/l_c \rightarrow \infty)$
$S_0$	$S_0$	$\frac{\nu_1 + \nu_2}{2} \frac{1 - e^{-L/l_c}}{\nu_2 - \nu_1 e^{-L/l_c}}$	0	$\frac{\nu_1 + \nu_2}{2 \max(\nu_1, \nu_2)}$
$S_0$	$S_1$	$\frac{\nu_1 + \nu_2}{2} \frac{1 + e^{-L/l_c}}{\nu_2 + \nu_1 e^{-L/l_c}}$	1	$\frac{\nu_1 + \nu_2}{2 \max(\nu_1, \nu_2)}$
$S_1$	$S_0$	$\frac{\nu_1 + \nu_2}{2} \frac{1 + e^{-L/l_c}}{\nu_2 + \nu_1 e^{-L/l_c}}$	1	$\frac{\nu_1 + \nu_2}{2 \max(\nu_1, \nu_2)}$
$S_1$	$S_1$	$\frac{\nu_1 + \nu_2}{2} \frac{1 - e^{-L/l_c}}{\nu_2 - \nu_1 e^{-L/l_c}}$	0	$\frac{\nu_1 + \nu_2}{2 \max(\nu_1, \nu_2)}$

$$\begin{bmatrix} J_1(x) \\ J_2(x) \end{bmatrix} = \frac{1}{\nu_2 - \nu_1} \times \begin{bmatrix} \nu_2 e^{-x/l_c} - \nu_1 & \nu_1 (1 - e^{-x/l_c}) \\ -\nu_2 (1 - e^{-x/l_c}) & \nu_2 - \nu_1 e^{-x/l_c} \end{bmatrix} \times \begin{bmatrix} J_1(0) \\ J_2(0) \end{bmatrix},$$

where

$$\frac{1}{l_c} = \frac{\sigma h}{e^2} \left( \frac{1}{\nu_1} - \frac{1}{\nu_2} \right). \quad (15)$$

If  $\sigma$  varies with  $x$ , the term  $x/l_c$  appearing in the exponentials in Eq. (15) has to be replaced by  $\frac{h}{e^2} (\frac{1}{\nu_1} - \frac{1}{\nu_2}) \int_0^x dx' \sigma(x')$ .

Now  $J_i(0)$  and  $J_i(L)$  are related to the incoming and outgoing currents  $I_i$  by the current splitting matrices at the ends of the LJ at  $x=0$  and  $L$ . Namely,

$$\begin{bmatrix} J_1(0) \\ I_4 \end{bmatrix} = S(0) \begin{bmatrix} I_1 \\ J_2(0) \end{bmatrix},$$

and

$$\begin{bmatrix} I_3 \\ J_2(L) \end{bmatrix} = S(L) \begin{bmatrix} J_1(L) \\ I_2 \end{bmatrix}. \quad (16)$$

Using Eqs. (15) and (16), the outgoing currents can be expressed in terms of the incoming currents as

$$\begin{pmatrix} I_3 \\ I_4 \end{pmatrix} = S_{LJ} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix},$$

where

$$S_{LJ} = \begin{pmatrix} 1 - \frac{2t\nu_2}{\nu_1 + \nu_2} & \frac{2t\nu_1}{\nu_1 + \nu_2} \\ \frac{2t\nu_2}{\nu_1 + \nu_2} & 1 - \frac{2t\nu_1}{\nu_1 + \nu_2} \end{pmatrix}, \quad (17)$$

and  $t$  is now the scattering coefficient of the LJ as a whole.

For the four different choices of  $S(0)$  and  $S(L)$  in terms of the two possible current splitting matrices  $S_0$  and  $S_1$  in Eq. (12), we find that the scattering coefficient  $t$  is given by the expressions in Table I. We see that depending on the choice of the matrices at the ends of the LJ,  $t$  lies in one of the two ranges  $\{0, (\nu_1 + \nu_2)/[2 \max(\nu_1, \nu_2)]\}$  and  $\{(\nu_1 + \nu_2)/[2 \max(\nu_1, \nu_2)], 1\}$ . Note that we need to have the non-trivial current splitting matrix  $S_1$  at one of the ends of the LJ in order to have  $t$  lie in the range  $\{(\nu_1 + \nu_2)/[2 \max(\nu_1, \nu_2)], 1\}$  where the system can act as a step-up transformer.



TABLE II. The scattering coefficient  $t$  for the four possible choices of  $S$  matrices at the ends of the LJ, for  $\nu_1 = \nu_2 = \nu$ .

$S(0)$	$S(L)$	$t$	$t(L \rightarrow 0)$	$t(L \rightarrow \infty)$
$S_0$	$S_0$	$\frac{\sigma L h / (\nu e^2)}{1 + \sigma L h / (\nu e^2)}$	0	1
$S_0$	$S_1$	1	1	1
$S_1$	$S_0$	1	1	1
$S_1$	$S_1$	1	1	1

For the special case  $\nu_1 = \nu_2 = \nu$ , we have to do a separate analysis of Eq. (14) since  $1/l_c = 0$ . Using the same procedure as described above, we find that the scattering coefficient is given by Table II.

We can relate the scattering coefficient  $t$  to the two-terminal conductance of the LJ as follows. Following Fig. 1(a), let us consider a situation in which  $V_2 = 0$  and therefore  $I_2 = 0$ . Equation (17) then implies that the current along the LJ,  $I_1 - I_4 = I_3$ , is related to the potential difference across the LJ,  $V_1 - V_2$ , as

$$G_{\text{LJ}} \equiv \frac{I_1 - I_4}{V_1 - V_2} = \frac{e^2}{h} \nu_1 \left( 1 - \frac{2t\nu_2}{\nu_1 + \nu_2} \right). \quad (18)$$

Thus a measurement of the conductance of the LJ,  $G_{\text{LJ}}$ , gives the value of  $t$ . For the special case  $\nu_1 = \nu_2 = \nu$ , this reduces to the expression  $G_{\text{LJ}} = (\nu e^2 / h) / [1 + \sigma L h / (\nu e^2)]$ , where we have used the first line of Table II; this agrees with the expression for the two-terminal conductance given in Ref. 5.

### B. Copropagating modes

We can repeat the above analysis for the case in which the two edges of the LJ propagate in the same direction as shown in Fig. 2(b). We now find that the currents satisfy the equations

$$\partial_x J_1 = -\partial_x J_2 = \frac{\sigma h}{e^2} \left( \frac{J_2}{\nu_2} - \frac{J_1}{\nu_1} \right). \quad (19)$$

Note that  $J_1(x) + J_2(x)$  is constant along the edge. If we assume the tunneling conductance  $\sigma$  to be independent of  $x$ , we obtain

$$\begin{bmatrix} J_1(x) \\ J_2(x) \end{bmatrix} = \frac{1}{\nu_2 + \nu_1} \times \begin{bmatrix} \nu_2 e^{-x/l_c} + \nu_1 & \nu_1 (1 - e^{-x/l_c}) \\ \nu_2 (1 - e^{-x/l_c}) & \nu_2 + \nu_1 e^{-x/l_c} \end{bmatrix} \begin{bmatrix} J_1(0) \\ J_2(0) \end{bmatrix},$$

where

$$\frac{1}{l_c} = \frac{\sigma h}{e^2} \left( \frac{1}{\nu_1} + \frac{1}{\nu_2} \right). \quad (20)$$

As before,  $J_i(0)$  and  $J_i(L)$  are related to the incoming and outgoing currents  $I_i$  by the current splitting matrices at  $x=0$  and  $L$ . We then find that the outgoing currents are again related to the incoming currents as in Eq. (17), where the scattering coefficient is given in Table III. The table remains valid for the special case  $\nu_1 = \nu_2$ .

The results given above are illustrated in Fig. 3 for the case  $\nu_1 = 1$  and  $\nu_2 = 1/3$ . We have shown the dependence of

 TABLE III. The scattering coefficient  $t$  for the four possible choices of  $S$  matrices at the ends of the LJ.

$S(0)$	$S(L)$	$t$	$t(L/l_c \rightarrow 0)$	$t(L/l_c \rightarrow \infty)$
$S_0$	$S_0$	$\frac{1 - e^{-L/l_c}}{2}$	0	$\frac{1}{2}$
$S_0$	$S_1$	$\frac{1 + e^{-L/l_c}}{2}$	1	$\frac{1}{2}$
$S_1$	$S_0$	$\frac{1 + e^{-L/l_c}}{2}$	1	$\frac{1}{2}$
$S_1$	$S_1$	$\frac{1 - e^{-L/l_c}}{2}$	0	$\frac{1}{2}$

the scattering coefficient  $t$  of the LJ on the dimensionless length  $L\sigma h/e^2$  for two different choices of the current splitting matrices at the ends of the LJ, for the cases of counter and copropagating edges. For the counter propagating case,  $t$  begins at 0 (or 1) for  $L \rightarrow 0$  and ends at  $(\nu_1 + \nu_2) / [2 \max(\nu_1, \nu_2)] = 2/3$  for  $L \rightarrow \infty$ . For the copropagating case,  $t$  begins at 0 (or 1) for  $L \rightarrow 0$  and ends at  $1/2$  for  $L \rightarrow \infty$ . In this picture, we have ignored the temperature dependence of  $\sigma$ . In Sec. VI, we will see how renormalization-group ideas can be used to study the temperature dependence of  $\sigma$  and therefore of  $t$ .

### V. DISORDER AND RENORMALIZATION GROUP

In this section, we will study the tunneling conductance  $\sigma(x)$  in more detail.<sup>5</sup> This arises from a tunneling amplitude  $\xi(x)$  appearing in a Hamiltonian density

$$\mathcal{H}_{\text{tun}} = \xi(x) \psi_1^\dagger(x) \psi_2(x) + h.c., \quad (21)$$

where  $\psi_i(x)$  denotes the electron annihilation operator at point  $x$  on edge  $i$  of the LJ. The tunneling conductance  $\sigma$  is then proportional to the tunneling probability  $|\xi|^2$ . It is believed that the presence of impurities near the LJ makes  $\xi(x)$  a random complex variable; let us assume it to be a Gaussian variable with a variance  $W$ . The quantity  $W$  satisfies an RG

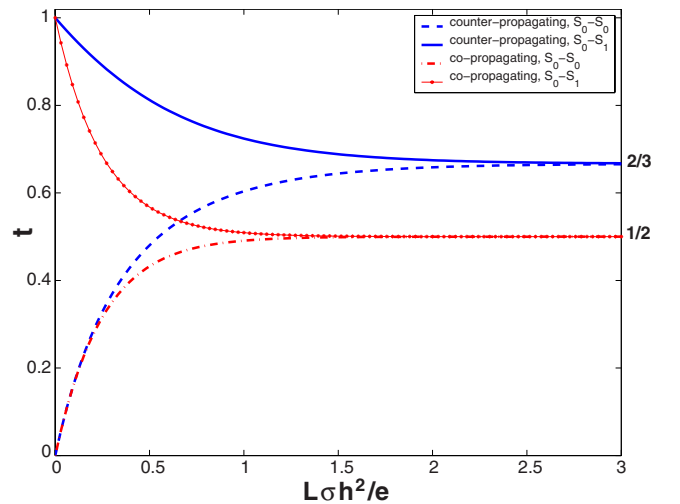


FIG. 3. (Color online) Scattering coefficient vs the dimensionless length of the line junction for two choices of the current splitting matrices at the ends  $x=0$  and  $L$ , for the cases of counter (blue lines) and copropagating (red lines) edges, with  $\nu_1 = 1$  and  $\nu_2 = 1/3$ .

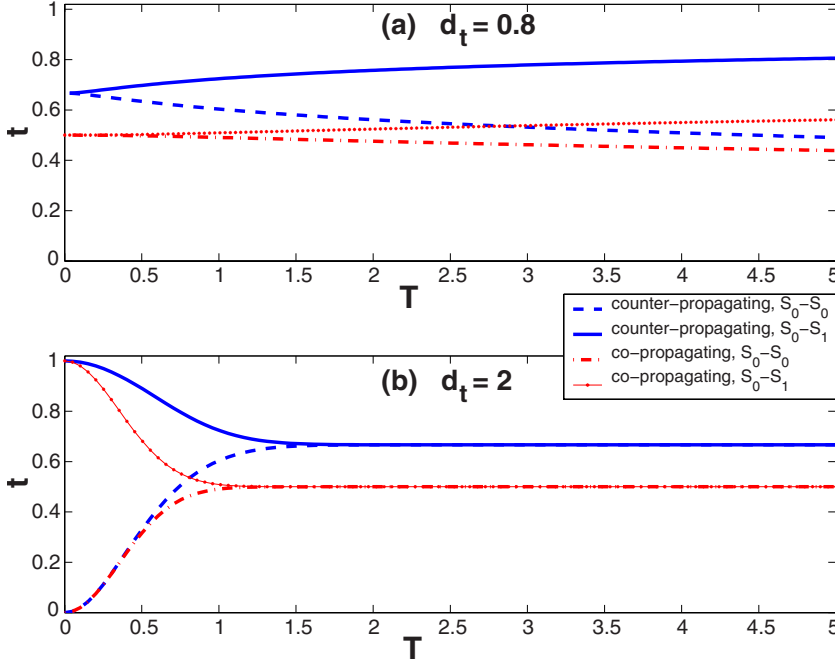


FIG. 4. (Color online) Scattering coefficient vs the dimensionless temperature for two choices of the current splitting matrices at the ends of the line junction, for the cases of counter (blue lines) and copropagating edges (red lines), with (a)  $d_t = 0.8$  and (b)  $d_t = 2$ , for  $\nu_1 = 1$  and  $\nu_2 = 1/3$ .

equation; to lowest order (i.e., for small  $\xi$ ), this equation is given by<sup>26,27</sup>

$$\frac{dW}{d \ln l} = (3 - 2d_t)W, \quad (22)$$

where  $l$  denotes the length scale, and  $d_t$  is the scaling dimension of the tunneling operator  $\psi_1^\dagger \psi_2$  appearing in Eq. (21). (We will calculate  $d_t$  below for both counter and copropagating cases). There is also an RG equation for the strength of the interaction between the electrons, but we can ignore that if  $W$  is small.

Let us first assume that the phase decoherence length  $L_T = \hbar v / (k_B T)$  (the length beyond which electrons lose phase coherence due to thermal smearing) is much smaller than the scattering mean free path  $L_m$  of the LJ. Successive back-scattering events then become incoherent and quantum interference effects of disorder are absent. One can then show that  $\sigma$  scales with the temperature  $T$  as  $T^{2d_t-2}$ .<sup>5</sup> (We note that  $\sigma$  is inversely proportional to the conductivity along the LJ studied in Ref. 5). It therefore seems that  $\sigma L \rightarrow 0$  as  $T \rightarrow 0$  if  $d_t > 1$ . However, it turns out that this is true only if  $d_t > 3/2$ , i.e., if  $W$  is an irrelevant variable according to Eq. (22). If  $d_t > 3/2$  (called the metallic phase), one can simultaneously have  $L \gg L_T$  (this is necessary to justify cutting off the RG flow at  $L_T$  rather than at  $L$ ), and  $\sigma L \rightarrow 0$ , i.e.,  $LT \gg 1$  and  $LT^{2d_t-2} \rightarrow 0$ , for some range of temperatures. Within this range, one can obtain the scattering coefficient  $t$  in Tables I–III by taking  $L/l_c \sim \sigma L \hbar / e^2 \rightarrow 0$ . If  $d_t < 3/2$  ( $W$  is a relevant variable), we have  $L/L_T \sim LT \gg 1$  and  $T^{2d_t-3} \rightarrow \infty$ ; hence  $\sigma L \sim LT^{2d_t-2} \rightarrow \infty$  (we call this the insulating phase). We can then obtain  $t$  in Tables I–III by taking  $L/l_c \sim \rightarrow \infty$ . We thus see that depending on whether  $d_t > 3/2$  or  $< 3/2$ , the parameter  $t$  tends to quite different values as the temperature is decreased.

The above analysis breaks down if one goes to very low temperatures where  $L_T \gtrsim L$  or  $L_m$ . In that case, the RG flow of  $W$  has to be cut off at the length scale  $L$  or  $L_m$ , rather than  $L_T$ ; hence  $\sigma$  and therefore the scattering coefficient  $t$  become independent of the temperature  $T$ .

In Fig. 4, we illustrate the temperature dependence of the scattering coefficient  $t$  for two choices of the current splitting matrices at the ends of the LJ, with  $d_t = 0.8$  and 2, for  $\nu_1 = 1$  and  $\nu_2 = 1/3$ . We have taken the conductance  $\sigma$  to scale as  $T^{2d_t-2}$  (specifically,  $L\sigma \hbar / e^2 = T^{2d_t-2}$ , where  $T$  is in dimensionless units), and then substituted that to obtain  $t$  from the first two rows of Tables I and III for counter and copropagating edges, respectively. For  $d_t < 0.8$  [Fig. 4(a)], we see that  $t$  approaches  $2/3$  ( $1/2$ ) as  $T \rightarrow 0$  for the counter (copropagating) cases, respectively, for any choice of the current splitting matrices at the ends of the LJ. For  $d_t = 2$  [Fig. 4(b)],  $t$  approaches 0 (1) as  $T \rightarrow 0$  depending on the choices of current splitting matrices at the ends of the LJ, regardless of whether the edges are counter or copropagating. As mentioned above, these pictures become invalid when we go to very low temperatures where  $L_T$  is not much smaller than  $L$  or  $L_m$ .

We will now compute the scaling dimension  $d_t$  of the operator  $\psi_1^\dagger \psi_2$  using bosonization. It is again convenient to discuss this for the cases of LJs with counter and copropagating modes separately.

### A. Counter propagating modes

For the LJ shown in Fig. 2(a), the mode on one edge goes from  $x=0$  to  $x=L$ , while the mode on the other edge goes in the opposite direction; let us call the corresponding bosonic fields  $\phi_1$  and  $\phi_2$ , respectively. In the absence of density-density interactions between these modes, the Lagrangian is given by

$$\begin{aligned} \mathcal{L} = & \frac{1}{4\pi\nu_1} \int_0^L dx \partial_x \phi_1 (-\partial_t - \nu_1 \partial_x) \phi_1 \\ & + \frac{1}{4\pi\nu_2} \int_0^L dx \partial_x \phi_2 (\partial_t - \nu_2 \partial_x) \phi_2, \end{aligned} \quad (23)$$

where  $\nu_i$  denotes the velocity of mode  $i$ . The bosonic fields can be expanded at time  $t=0$  as

$$\phi_1 = \int_0^\infty \frac{dk}{k} (b_1 e^{ikx} + b_1^\dagger e^{-ikx}),$$

and

$$\phi_2 = \int_0^\infty \frac{dk}{k} (b_2 e^{-ikx} + b_2^\dagger e^{ikx}), \quad (24)$$

where the creation and annihilation operators satisfy the commutation relations

$$[b_{ik}, b_{jk'}^\dagger] = \delta_{ij} \nu_i k \delta(k - k'). \quad (25)$$

The electron annihilation operator on edge  $i$  is given by  $\chi_i e^{i\phi_i/\nu_i}$ ,  $\chi_i$  being the electron Klein factor. The tunneling operator between the edges,  $\psi_1^\dagger \psi_2$ , is therefore given by  $\chi_1^\dagger \chi_2 e^{i(\phi_2/\nu_2 - \phi_1/\nu_1)}$ . (Since the edges belong to different QH systems, quasiparticles with fractional charge cannot tunnel between the two edges as that would change the charge of each QH system by a fractional amount).

We will again assume that  $L_T \ll L_m \ll L$ ; therefore, two points on the LJ which are separated by a distance much larger than  $L_T$  are not related to each other in a phase-coherent way. In particular, at all points deep inside the LJ, i.e., separated from the ends of the LJ at  $x=0$  and  $L$  by a distance much larger than  $L_T$ , the bosonic fields carry no information about the current splitting matrices  $S$  appearing at the ends. We can therefore assume that  $\phi_1(x)$  and  $\phi_2(x)$  are independent fields at all points  $x$  except points very close to the edges. For the same reason, we can replace the limits of the integration in Eq. (23) by  $-\infty$  and  $\infty$  since only fields lying within a distance of about  $L_T$  from a given point  $x$  will contribute to tunneling at that point. We can now read off the scaling dimension of the tunneling operator from the Lagrangian in Eq. (23); we find that  $d_t = (1/2)(1/\nu_1 + 1/\nu_2)$ . For instance, for a LJ lying between QH systems with  $\nu_1, \nu_2$  equal to 1 and  $1/3$ ,  $d_t=2$  which means that the disorder parameter  $W$  is irrelevant.

We will now consider the effect of a short-range density-density interaction between the two edges on the scaling dimension  $d_t$ . The densities for the two modes are given by  $\rho_1 = (1/2\pi)\partial_x \phi_1$  and  $\rho_2 = -(1/2\pi)\partial_x \phi_2$ , respectively. Hence a repulsive interaction will correspond to a term in the Lagrangian of the form

$$\mathcal{L}_{\text{int}} = \frac{\lambda}{4\pi\sqrt{\nu_1\nu_2}} \int_0^L dx \partial_x \phi_1 \partial_x \phi_2, \quad (26)$$

where  $\lambda$  is a positive number with the dimensions of velocity. The Hamiltonian corresponding to Eqs. (23) and (26) is then given by

$$H = \int_0^\infty dk \left[ \frac{\nu_1}{\nu_1} b_{1k}^\dagger b_{1k} + \frac{\nu_2}{\nu_2} b_{2k}^\dagger b_{2k} - \frac{\lambda}{2\sqrt{\nu_1\nu_2}} (b_{2k}^\dagger b_{1k}^\dagger + b_{1k} b_{2k}) \right]. \quad (27)$$

This can be diagonalized by a Bogoliubov transformation. We then obtain new bosonic fields  $\tilde{\phi}_1$  and  $\tilde{\phi}_2$  which have the velocities

$$\tilde{\nu}_1 = \frac{1}{2} [\sqrt{(\nu_1 + \nu_2)^2 - \lambda^2} + \nu_1 - \nu_2],$$

and

$$\tilde{\nu}_2 = \frac{1}{2} [\sqrt{(\nu_1 + \nu_2)^2 - \lambda^2} + \nu_2 - \nu_1]. \quad (28)$$

The requirement of stability,  $\tilde{\nu}_1, \tilde{\nu}_2 > 0$ , means that we must have  $4\nu_1\nu_2 > \lambda^2$ . Finally, we can obtain the scaling dimension of the tunneling operator  $e^{i(\phi_2/\nu_2 - \phi_1/\nu_1)}$  after rewriting  $\phi_i$  in terms of the new fields  $\tilde{\phi}_i$ . We discover that

$$d_t = \frac{1}{4K} \left[ (1 + K^2) \left( \frac{1}{\nu_1} + \frac{1}{\nu_2} \right) - \frac{2(1 - K^2)}{\sqrt{\nu_1\nu_2}} \right],$$

where

$$K = \sqrt{\frac{\nu_1 + \nu_2 - \lambda}{\nu_1 + \nu_2 + \lambda}}. \quad (29)$$

For the special case  $\nu_1 = \nu_2 = \nu$ , Eq. (29) gives  $d_t = K/\nu$ ,<sup>5</sup> while for  $K=1$ , we get  $d_t = 1/(2\nu_1) + 1/(2\nu_2)$ . It is interesting to note that for a given value of  $\nu_1$  and  $\nu_2$ ,  $d_t$  has a nonmonotonic dependence on  $K$ . When  $K$  is reduced from 1 by turning on a weak repulsive interaction (i.e.,  $\lambda$  is small and positive),  $d_t$  starts decreasing; however,  $d_t$  reaches a minimum at  $K = |\sqrt{\nu_1 - \sqrt{\nu_2}}| / (\sqrt{\nu_1} + \sqrt{\nu_2})$ , beyond which it starts increasing as  $K$  decreases further.

## B. Copropagating modes

For the LJ shown in Fig. 2(b), the modes on both edges go from  $x=0$  to  $x=L$ . Hence both modes have a Lagrangian and an expansion similar to that of the field  $\phi_1$  given in Eqs. (23) and (24). In the presence of density-density interactions, the Hamiltonian is given by

$$H = \int_0^\infty dk \left[ \frac{\nu_1}{\nu_1} b_{1k}^\dagger b_{1k} + \frac{\nu_2}{\nu_2} b_{2k}^\dagger b_{2k} - \frac{\lambda}{2\sqrt{\nu_1\nu_2}} (b_{2k}^\dagger b_{1k}^\dagger + b_{1k} b_{2k}) \right]. \quad (30)$$

This can be diagonalized by a simple rotation. The new bosonic fields have the velocities

$$\tilde{\nu}_1 = \frac{1}{2} [\nu_1 + \nu_2 + \sqrt{(\nu_1 - \nu_2)^2 + \lambda^2}],$$

and

$$\tilde{\nu}_2 = \frac{1}{2}[v_1 + v_2 - \sqrt{(v_1 - v_2)^2 + \lambda^2}]. \quad (31)$$

Once again stability, i.e.,  $\tilde{\nu}_2 > 0$ , requires that  $4v_1v_2 > \lambda^2$ . Finally, the scaling dimension of the tunneling operator  $e^{i(\phi_2/\nu_2 - \phi_1/\nu_1)}$  is found to be given by

$$d_t = \frac{1}{2\nu_1} + \frac{1}{2\nu_2}, \quad (32)$$

independent of the strength of the interaction  $\lambda$ .

## VI. EXPERIMENTAL IMPLICATIONS

Our results can be experimentally tested in bent QH systems such as the ones studied in Refs. 1 and 2. A gate voltage can be used to control the distance between the two edges of the LJ. Making the gate voltage less repulsive for electrons is expected to reduce the distance between the edges; this should increase both the strength of the density-density interactions as well as the tunneling conductance.<sup>5</sup> In this way, one may be able to vary the scaling dimension  $d_t$  across the value  $3/2$ .

For the case of counter propagating edges with  $\nu_1 \neq \nu_2$ , we have discussed in Sec. IV A how  $t$  can be obtained from a two-terminal conductance measurement. Our first observation is that the value of  $t$  always lies in one of two mutually exclusive ranges,  $\{0, (\nu_1 + \nu_2)/[2 \max(\nu_1, \nu_2)]\}$  or  $\{(\nu_1 + \nu_2)/[2 \max(\nu_1, \nu_2)], 1\}$ ; this can be seen in Table I. Next, we saw in Sec. V that for  $d_t > 3/2$ ,  $t$  will approach either 0 or 1, depending on which of the two ranges  $t$  happens to lie in, as the temperature is lowered (provided that  $L_T \ll L$ ). On the other hand, if  $d_t < 3/2$ ,  $t$  will approach the value  $(\nu_1 + \nu_2)/[2 \max(\nu_1, \nu_2)]$  from below or above, depending on which of the two ranges  $t$  lies in, as the temperature is lowered. Finally, the rate at which the various asymptotic values of  $t$  is approached depends on the value of  $d_t$ ; this value is determined by  $\nu_1$ ,  $\nu_2$  and the interaction strength  $\lambda$  which can be controlled by the gate voltage.

For the case of copropagating edges with any values of  $\nu_1$  and  $\nu_2$ , the value of  $t$  always lies in one of two mutually exclusive ranges,  $[0, 1/2]$  or  $[1/2, 1]$ ; this can be seen in Table III. For  $d_t > 3/2$ ,  $t$  will approach either 0 or 1, depending on which of the two ranges  $t$  happens to lie in, as the temperature is lowered. If  $d_t < 3/2$ ,  $t$  will approach the value  $1/2$  from below or above as the temperature is lowered. Unlike the case of counter propagating edges, the rate at which the various asymptotic values of  $t$  are approached now depends on only  $\nu_1$  and  $\nu_2$ , and not on the interaction strength or the gate voltage.

In the presence of interactions and disorder,  $L_m$  scales with temperature as  $T^{2-2d_t}$  and  $L_T \sim T^{-1}$ .<sup>5</sup> Thus throughout the metallic phase ( $d_t > 3/2$ ),  $L_m \gg L_T$  as  $T \rightarrow 0$ . We note again that this is the regime of validity of our analysis.

## VII. DISCUSSION

In this work we have developed a model for studying transport along a QH line junction with either counter and

copropagating modes, in the case of QH states for which each edge has a single chiral mode. Each end of the line junction is described by a current splitting matrix whose form is severely restricted by the requirement that the bosonic fields at those points should be linearly related to each other. We then consider the effect of tunneling across all points of the line junction and obtain expressions for the current splitting matrix  $S_{LJ}$  of the line junction in terms of the filling fractions, the tunneling conductance and the length of the line junction. Next, the tunneling conductance is taken to be a random variable; its temperature dependence is obtained using renormalization-group ideas. The scaling dimension of the tunneling operator is found to depend on the strength of the interaction between the two edges of the line junction in the counter propagating case, but not in the copropagating case. Depending on the scaling dimension, the system can exhibit two different behaviors as the temperature is decreased. For a line junction with counter propagating modes, one can change the behavior by applying a gate voltage placed above the line junction since such a voltage can change the effective width of the line junction and therefore the strength of the interactions. Our model provides a theoretical framework for analyzing experimental studies of the transport properties of line junctions in QH systems; we have discussed some experimental implications of our results.

It would be useful to extend the analysis presented in this paper to the regime of very low temperature where  $L_T \gtrsim L$  and  $L_m$ . There are two problems which need to be addressed in order to do this. First, the kinetic equation approach used in Sec. III needs to be modified in this regime since that approach implicitly assumes that the phase decoherence length is much smaller than the scattering mean free path of the LJ. Second, the bosonic fields on the two edges of the LJ are not independent of each other at low temperatures if the current splitting matrices at the ends of the LJ are taken to be of the form  $S_1$ , since such a matrix mixes the bosonic fields on all the incoming and outgoing edges if  $\nu_1 \neq \nu_2$ .

It would also be interesting to extend our analysis to the case of QH systems with more complicated filling fractions whose edges are described by multiple modes, some of which may propagate in opposite directions.<sup>24,28-30</sup>

Finally, we would like to mention studies of a QH system with a point-contact interface separating two different filling fractions,<sup>31</sup> and a QH system with an extended constriction with the same filling fraction on the two sides.<sup>32</sup> It may be possible to extend our analysis to these systems as well.

After this work was submitted for publication, a paper has appeared which considers a system with several line junctions, each of which separates quantum Hall fluids with different filling fractions.<sup>33</sup> Their result for the conductance of a single-line junction agrees with ours.

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